Non-existence of free S^1-actions on Kervaire spheres in dimensions less than 130 (Transformation Group Theory and Surgery)

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Non-existence of free S^1-actions on Kervaire spheres in dimensions less than 130

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The purpose of this note is to show that the Kervaire spheres Σ_K^{4k+1} where $4k+4$ is not a power of 2, does not admit any free S^1-action provided $k \leq 32$. Recall here that the Kervaire sphere Σ_K^{4k+1} is a homotopy sphere and it is a boundary of a parallelizable $4k+2$ manifold W^{4k+2} with Kervaire invariant $c(W)=1$. The explicit description of the Kervaire sphere is possible and one of the simplest one is given by the system of equations

$$
\begin{aligned}
&z_1^d + z_2^2 + \cdots + z_{2k+2}^2 = 0 \\
&|z_1|^2 + |z_2|^2 + \cdots + |z_{2k+2}|^2 = 1
\end{aligned}
$$

in $\mathbb{C}^{2k+2}$, where d is any positive integer such that $d \equiv \pm 3 \mod 8$. Those homotopy spheres that bound parallelizable manifolds are considered to be "least" exotic among exotic spheres. Not only are they represented by simple formulas, but they are abundant in symmetry; the dimension of Lie groups that can effectively act on them are known to be high. As to free actions of finite groups, Kervaire spheres have free actions of finite cyclic groups of arbitrary order. Odd dimensional standard spheres S^{2n+1} has free S^1-actions and so it may seem natural to expect that Kervaire spheres also have free S^1-actions. But the case is quite different as to the free action of Lie groups. In fact, more than thirty years ago, Brumfiel showed that 9-dimensional Kervaire sphere do not have any free S^1-actions. For a long period of time since then, this problem has been left untouched. The reason seems simple. Brumfiel's calculation can be extended to other higher dimensions at least theoretically, and with the aid of computer, we can actually perform this computation. However the resulting relation is not linear and it seems extremely hard to draw effective and simple conclusions from the those data.

In last year's workshop, we did similar calculation to the one by Brumfiel and proved that the Kervaire sphere Σ_K^{17} does not admit any free S^1-action. It is not wise to continue this calculation because as dimension gets larger, the result of the computation gets uncontrollably longer. In this paper, by taking modulo 2^r in all calculations for an appropriate integer r, we succeeded in solving the conjecture in dimensions less than 130.

Our main result is the following:

Theorem. The Kervaire sphere in dimensions less than 130 (except for dimensions 5, 13, 29, 61, 125) does not admit any free S^1-actions.

We shall assume that k stands for a positive integer such that $k+1$ is not a power of two. Then it is known that the Kervaire sphere Σ_K^{4k+1} is not diffeomorphic to the standard sphere S^{4k+1}.

1. Surgery Obstruction

If the Kervaire sphere Σ_K^{4k+1} admits a free S^1-action, the quotient space of the S^1-action $X^{4k} = \Sigma^{4k+1}/S^1$ is homotopy equivalent to the complex projective space $\mathbb{C}P(2k)$ and the associated D^2-bundle $N^{4k+2} = (\Sigma_K^{4k+1} \times D^2)/S^1$ is homotopy equivalent to $(S^{4k+1} \times D^2)/S^1$ where the $S^1 \subset \mathbb{C}$ acts on $S^{4k+1} \subset \mathbb{C}^{2k+1}$ and on $D^2 \subset \mathbb{C}$ by complex number multiplication. Let W^{4k+2} be a smooth parallelizable manifold with $\partial W = \Sigma_K^{4k+1}$ and Kervaire invariant $c(W) = 1$. Then by gluing N and W along the common boundary Σ_K, we obtain a normal map $f : P = N \cup_{\Sigma_K} W \to \mathbb{C}P(2k+1)$ with appropriate bundle data, and its surgery obstruction s_{4k+2} is equal to $c(W) = 1$. Hence we have a normal map f with target space $\mathbb{C}P(2k+1)$ with nonzero Kervaire surgery obstruction, but the codimension 2 surgery problem obtained by restricting the target manifold to $\mathbb{C}P(2k)$ has zero surgery obstruction $s_{4k} = 0$.

Conversely if we are given a normal map $f : P \to \mathbb{C}P(2k+1)$ such that the surgery obstruction s_{4k+2} of f is nonzero and the restricted surgery problem to $\mathbb{C}P(2k)$ has zero surgery obstruction $s_{4k} = 0$. Then we can perform surgery on $f^{-1}(\mathbb{C}P(2k))$ and within the normal cobordism class we may assume that $X = f^{-1}(\mathbb{C}P(2k)) \to \mathbb{C}P(2k)$ is a homotopy equivalence. The tubular neighborhood N of X is homotopy equivalent to $\mathbb{C}P(2k+1)_0$ and the ∂N is homotopy equivalent to S^{4k+1}. But the remaining part $W = P - \mathrm{int}(N)$ is a parallelizable manifold and its surgery obstruction for the normal map $W \to D^{4k+2}$ is nonzero. Therefore W has nonzero Kervaire obstruction and therefore its boundary is the Kervaire sphere.

Proposition 1. The following two statements are equivalent.
(a) The Kervaire sphere Σ_K^{4k+1} does not admit any free S^1-action.
(b) If the normal map $f : M^{4k+2} \to \mathbb{C}P(2k+1)$ with appropriate bundle data

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M^{4k+2} & \xrightarrow{f} & \mathbb{C}P(2k+1) \end{array} \tag{1}$$

has zero $4k$-dimensional surgery obstruction $s_{4k} = 0$ for the surgery data

$$f | f^{-1}(\mathbb{C}P(2k)) : f^{-1}(\mathbb{C}P(2k)) \to \mathbb{C}P(2k)$$

then the $(4k+2)$-dimensional surgery obstruction s_{4k+2} of f vanishes.

Our objective of this note is to show that the statement (b) in Proposition 1 is true. To do so, we must deal with all possible bundle data that appear in (1). We point out the following four items that needs consideration:

Bundle data: The stable bundle difference $\zeta = \nu_{\mathbb{C}P(2k+1)} - \xi$ is fiber homotopically trivial, namely it belongs to the kernel of the J-homomorphism $J : \widetilde{KO}(\mathbb{C}P(2k+1)) \to \tilde{J}(\mathbb{C}P(2k+1))$. The generators of the kernel can be expressed by Adams operations in KO-theory.

The surgery obstruction s_{4k} in dimension $4k$: In dimension $4k$, the surgery obstruction is given by the index obstruction, which can be computed using Hirzebruch's L classes. However, the exact form of the obstruction gets complicated and requires simplified treatment.

Surgery obstruction s_{4k+2} in dimension $4k+2$: The surgery obstruction s_{4k+2} in dimension $4k+2$ can be dealt with by the results of Stolz([5]) and Kitada([3],[4]). In fact, the obstruction s_{4k+2} is equal to the two dimensional obstruction s_2 for the surgery data s_2, which is essentially the 2-dimensional Kervaire class K_2.

Relation of K_2 and the first Pontrjagin class p_1: From the result originally due to Sullivan, the square of K_2 for the bundle data ζ is equal to $p_1(\zeta)/8 \mod 2$. This fact gives the connection of the integral index obstruction and the mod 2 Kervaire obstruction.

2. Index obstruction in dimension $4k$

The kernel of the 2-local J-homomorphism $J : \widetilde{KO}(\mathbb{C}P(2k+1)) \to \tilde{J}(\mathbb{C}P(2k+1))$ is generated by Image$(\psi^q_{\mathbb{R}}-1)$ (q odd), where $\psi^q_{\mathbb{R}}$ is the Adams operation in KO-theory and we may assume that $q = 3$. The additive generators of $\widetilde{KO}(\mathbb{C}P(2k+1))$ are given by ω^j $(1 \leq j \leq k+1)$ where ω is the realification of the complex virtual vector bundle $\eta_{\mathbb{C}} - 1_{\mathbb{C}}$. The Adams operation $\psi^j_{\mathbb{R}}$ on ω is given by the formula

$$\psi^j_{\mathbb{R}}(\omega) = T_j(\omega) \tag{2}$$

where $T_j(z)$ is a polynomial of degree j characterized by

$$T_j(t + t^{-1} - 2) = t^j + t^{-j} - 2\,. \tag{3}$$

Since the coefficient of z^j in $T_j(z)$ is one, we may consider $T_j(\omega)$ $(1 \leq j \leq k+1)$ as generators of $\widetilde{KO}(\mathbb{C}P(2k+1))$. However, when restricted on $\mathbb{C}P(2k)$, we have $\omega^{k+1} = 0$ and we may safely discard ω^{k+1} in the actual computation. In our argument, we do not necessarily need to know the kernel of $J : \widetilde{KO}(\mathbb{C}P(2k+1)) \to \tilde{J}(\mathbb{C}P(2k+1))$. Later computation shows that we can ignore odd multiples of elements and we have only to know 2-local generators of the kernel. The 2-local generators of the kernel of J is generated by

$$\zeta_j = (\psi^3_{\mathbb{R}} - 1)\psi^j_{\mathbb{R}}(\omega) \qquad (j = 1, 2, \ldots, k) \tag{4}$$

and an element of the 2-local kernel of the J-homomorphism has the form

$$\zeta = \sum_{j=1}^{k} m_j \zeta_j \tag{5}$$

where m_j are integers.

The surgery obstruction s_{4k} of the surgery data (1) when restricted on $\mathbb{C}P(2k)$ is given by

$$8s_{4k} = (\mathrm{Index}(M) - \mathrm{Index}(\mathbb{C}P(2k))) = ((\mathcal{L}(\zeta) - 1)\mathcal{L}(\mathbb{C}P(2k)))\,[\mathbb{C}P(2k)] \tag{6}$$

where $\mathcal{L}$ is the multiplicative class defined by the power series

$$h(x) = \frac{x}{\tanh x} = 1 + \sum_{i \geq 1} \frac{(-1)^{i+1} 2^i B_i}{(2i)!} x^{2i} \tag{7}$$

where B_i is the i-th Bernoulli number. Remark that all the coefficients of $h(x)$ belong to $\mathbb{Z}_{(2)}$ the rational numbers with odd denominator because all the denominators of Bernoulli numbers are even but not divisible by four. If the total

Pontrjagin class of a bundle ξ is given by $p(\xi) = \prod_i (1 + x_i^2)$, $\mathcal{L}(\xi)$ is given by $\prod_i h(x_i)$ and when M is a manifold, we define $\mathcal{L}(M) = \mathcal{L}(\tau_M)$.

It is not difficult to show that the total Pontrjagin class of $\psi^j_{\mathbb{R}}(\omega)$ is $1 + j^2x^2$, where x is the generator of $H^2(\mathbb{C}P(2k+1))$. For the virtual bundle ζ in (5), we have

$$\mathcal{L}(\zeta) = \prod_{j=1}^{k} \left(\frac{3jx}{\tanh 3jx} \frac{\tanh jx}{jx} \right)^{m_j} . \tag{8}$$

Given a power series $f(x)$ in x, we shall write the coefficient of x^n in $f(x)$ by $(f(x))_n$. The $4k$-dimensional obstruction s_{4k} is given by

$$\left((\mathcal{L}(\zeta) - 1) \left(\frac{x}{\tanh x} \right)^{2k+1} \right)_{2k} / 8. \tag{9}$$

To calculate this, we put

$$g(x) = \left(\frac{3x}{\tanh 3x} \frac{\tanh x}{x} \right) - 1. \tag{10}$$

Lemma 2. All the coefficients of $g(x)$ belong to $8\mathbb{Z}_{(2)}$.

From the expansion (7), we have

$$\frac{3x}{\tanh 3x} \equiv \frac{x}{\tanh x} \quad \bmod 8 \quad \text{in} \quad \mathbb{Z}_{(2)}[[x]]$$

and the assertion follows.

We now calculate the $\mathcal{L}$ class:

$$\begin{aligned} \mathcal{L}(\zeta) - 1 &= \prod_j (1 + g(jx))^{m_j} - 1 \\ &= \prod_j \left(1 + m_j g(jx) + \frac{m_j(m_j-1)}{2} (g(jx))^2 + \cdots \right) - 1 \end{aligned}$$

From this, if we want to calculate the $4k$-dimensional surgery obstruction s_{4k} mod 8, we can get it from

$$\begin{aligned} (\mathcal{L}(\zeta) - 1) h(x)^{2k+1} &\equiv h(x)^{2k+1} \prod_j ((1 + m_j g(jx)) - 1) \quad \bmod 64 \\ &\equiv h(x)^{2k+1} (\sum_j m_j g(jx)) \qquad \bmod 64 \end{aligned}$$

and if we want to calculate mod 64 value of the obstruction we may get it from

$$\begin{aligned} & h(x)^{2k+1} \prod_j \left(\left(1 + m_j g(jx) + \frac{m_j(m_j-1)}{2} (g(jx))^2 \right) - 1 \right) \\ & \equiv h(x)^{2k+1} \left(\sum_j \left(m_j g(jx) + \frac{m_j(m_j-1)}{2} (g(jx))^2 \right) \right. \\ & \qquad \left. + \sum_{i<j} (m_i m_j g(ix) g(jx)) \right) \quad \bmod 64. \end{aligned}$$

Thus we may obtain $4k$-dimensional surgery obstruction formula mod 8 or mod 64 in terms of integers m_j. If this obstruction vanishes, we obtain a relation among the integers m_j.

3. First Pontrjagin class and Kervaire surgery obstruction

In the normal map (1), let $\zeta = \nu_{\mathbb{C}P(2k+1)} - \xi$, then it can be written (2-locally) $\zeta = \sum_{j=1}^{k} m_j \zeta_j$ where $\zeta_j = (\psi_{\mathbb{R}}^{3} - 1)\psi_{\mathbb{R}}^{j}(\omega)$. The total Pontrjagin class of $\psi_{\mathbb{R}}^{m}(\omega)$ is given by

$$p(\psi_{\mathbb{R}}^{m}(\omega)) = 1 + m^2x^2$$

and we have

$$p(\zeta_j) = \frac{1+9j^2x^2}{1+j^2x^2}$$

$$p(\zeta) = \prod_j \left(\frac{1+9j^2x^2}{1+j^2x^2}\right)^{m_j}.$$

For the first Pontrjagin class, we have

$$p_1(\zeta)/8 = \left(\sum_j j^2 m_j\right) x^2. \tag{11}$$

We know that the 2-dimensional surgery obstruction s_2 for $f|f^{-1}(\mathbb{C}P(1))$ is equal to $\sum_j j^2 m_j \mod 2$ since in the complex projective space surgery theory, the mod 2 reduction of $p_1(\zeta)$ coincides with the square of the 2-dimensional Kervaire class for the given normal map (see Wall's book [6, Chap 13.]). And it is known that if $k+1$ is not a power of 2, then $(4k+2)$-dimensional surgery obstruction coincides with 2-dimensional surgery obstruction ([5],[3],[4]). From these facts we have

Lemma 3. If $\sum_{j:odd} m_j$ is even, then the surgery obstruction s_{4k+2} vanishes.

In fact, we are able to show that if the $4k$-dimensional surgery obstruction s_{4k} vanishes, then $\sum_{j:odd} m_j$ is even in dimensions less than 130, i.e. for $k \leq 32$ and $k+1$ not a power of two. We shall present some typical cases of our computation in the next last section.

4. Some examples of computation

Case $k = 2$: (Brumfiel's original case)

$$\left(\left(\prod_{j=1}^{2}\left(\frac{3jx}{\tanh 3jx}\frac{\tanh jx}{jx}\right) - 1\right)\left(\frac{x}{\tanh x}\right)^5\right)_4 \equiv 16m_1 \mod 64$$

If this is zero, then m_1 must be even.

Case $k = 4$: (Last year's computation)

$$\left(\left(\prod_{j=1}^{4}\left(\frac{3jx}{\tanh 3jx}\frac{\tanh jx}{jx}\right) - 1\right)\left(\frac{x}{\tanh x}\right)^9\right)_8 \equiv 32m_1 + 32m_3 \mod 64.$$

From this we have $\sum_{j:odd} m_j = m_1 + m_3$ is even.

Case $k=7$:

$$\left(\left(\prod_{j=1}^{8}\left(\frac{3jx}{\tanh 3jx}\frac{\tanh jx}{jx}\right)-1\right)\left(\frac{x}{\tanh x}\right)^{15}\right)_{14} \equiv 8m_1+8m_3+8m_5+8m_7 \mod 64.$$

From this we have $m_1+m_3+m_5+m_7$ is even.

The computation breaks down for $k=8$, because we obtain no information from mod 64 computation. We should, instead, perform computation mod 512 for k divisible by 8.

Case $k=8$:

$$\begin{aligned}&\left(\left(\prod_{j}^{8}\left(\frac{3jx}{\tanh 3jx}\frac{\tanh jx}{jx}\right)-1\right)\left(\frac{x}{\tanh x}\right)^{17}\right)_{16}\\ &\equiv 128(m_1^2+m_3^2+m_5^2+m_7^2)\\ &+256(m_1m_3+m_1m_5+m_1m_7+m_3m_5+m_3m_7+m_5m_7)\\ &+192m_1+256m_2-64m_3-128m_4-64m_5+256m_6+192m_7 \mod 512\end{aligned}$$

This is equivalent to

$$64(m_1+m_3+m_5+m_7) \mod 128$$

So if this vanishes, then $\sum_{j:odd} m_j$ should be even. For $k \leq 32$ not a multiple of 8, computation mod 64 solves our problem and for $k \leq 32$ which is a multiple of 8, the computation mod 512 solves our problem too. Namely, in either case, for $k \leq 32$, if $s_{4k}=0$, then $\sum_{j:odd} m_j$ must be even and we have $s_{4k+2}=0$.

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